Probing p-adic CFT using p-adic bulk & PGL_2

Hao-Yu Sun
University of Texas, Austin, USA

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arXiv:1911.06313, w/ Stephen Ebert (UCLA) & Meng-Yang Zhang (Princeton)

Motivation

- To get a **discretized** spacetime model which is most similar to ordinary AdS spacetime (unlike e.g. tensor networks).
  Many techniques carry over: propagators, Witten diagrams, GKP-W dictionary, etc…

- Inspired by the success of p-adic string in the 80s [Brekke, Freund, Olson, Witten; Vladimirov; Dragović, Volovich; many others…]

  Veneziano & Shapiro-Virasoro amplitudes: adelic product

- Connections with strongly-correlated condensed matter systems: e.g. ultra-cold atoms [Bentsen, Hashizume, Buyskikh, Davis, Daley, Gubser, Schleier-Smith, '19]

- Earlier work on connections between dS/CFT and BT tree in the context of eternal inflation [Harlow, Shenker, Stanford, Susskind, '12]

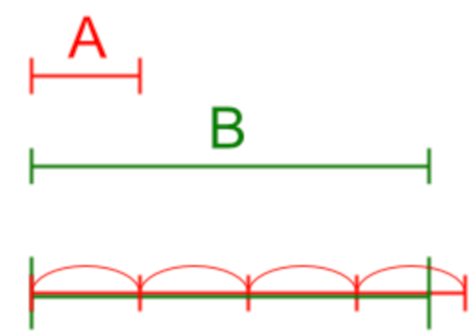
- Modern foundation: [Gubser, Knaute, Parikh, Samberg, Witaszczyk, '16]
  [Heydeman, Marcolli, Saberi, Stoica, '16]

1. Introduction to p-adic

A (not so) weird # system

The tortoise can't take over the hare!

- Non-Archimedean: $\sup\{|n|_{\mathbb{F}} : n \in \mathbb{Z}_{\mathbb{F}}\} = 1$, not $\sup\{|n|_{\mathbb{F}} : n \in \mathbb{Z}_{\mathbb{F}}\} = \infty$



- $\Rightarrow$ Ultrametricity: $|x+y|_{\mathbb{F}} \leq \sup\left(|x|_{\mathbb{F}}, |y|_{\mathbb{F}}\right)$
  $\Rightarrow$ All triangles are "tall isosceles", otherwise triangle ineq/axiom $|x+y|_{\mathbb{F}} \leq |x|_{\mathbb{F}} + |y|_{\mathbb{F}}$ violated

Basic idea: 1. View p as a small but nonzero #

2. All integers coprime to p have the same size

- For a prime p, $\mathbb{Q}_p$ is a completion of rationals $\mathbb{Q}$ w.r.t. the p-adic norm $|\cdot|_p$
  Any $x \in \mathbb{Q}_p$ has a **unique** p-adic expansion:

$$x = \underbrace{...a_3a_2a_1a_0}_{\text{in } \mathbb{Z}_p} . \underbrace{a_{-1}a_{-2}...a_{v_p}}_{\text{fractional part of } x} \equiv \sum_{n=v_p}^{\infty} a_n p^n$$

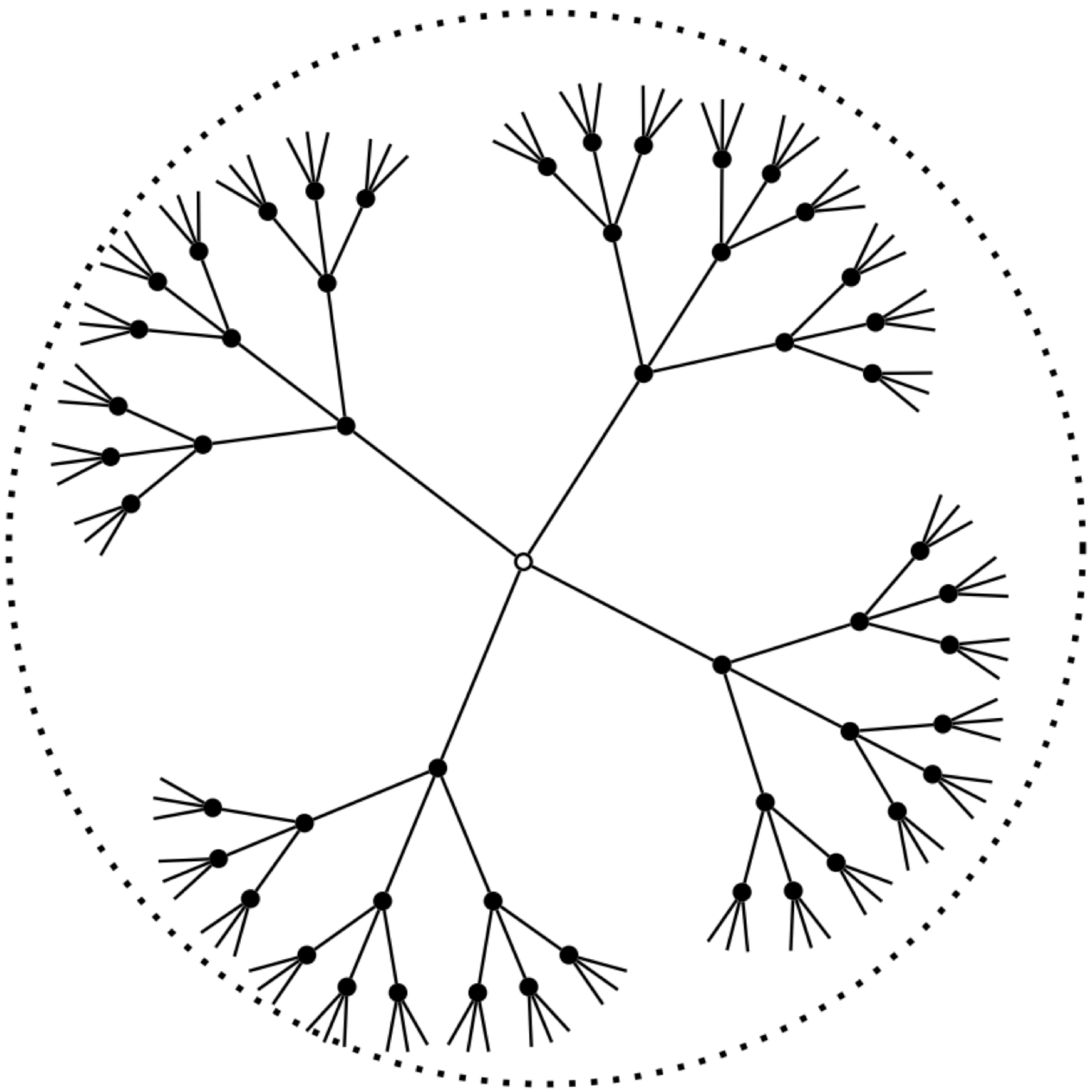
From right to left! $\longleftarrow$

  where $a_n \in \{0,1,\cdots,p-1\}$, and v_p is the smallest integer index s.t. $a_{v_p} \neq 0$
- Norm $|x|_p = p^{-v_p}$

- Ostrowski's theorem: $\mathbb{Q}$ only has two completions! $\mathbb{R}$ & $\mathbb{Q}_p$

Bruhat-Tits (BT) tree

- A great visualization of p-adic #'s
- T_p has valence $p+1$, also called "Bethe lattice", such as in [Baxter, '82]
- 1st introduced into hep-th by [Zabrodin, '89] as the interior of an open-string worldsheet
- $\partial T_p = P^1(\mathbb{Q}_p) \equiv \mathbb{Q}_p \cup \infty$
- Isometry group is $PGL(2, \mathbb{Q}_p)$
- Key player for p-adic holography
- Straightforward for unramified extension $\mathbb{Q}_{p^n}$, an n-dim vector space over $\mathbb{Q}_p$
- C.f. talk by Ling-Yan (Janet) on Tuesday and the talk by Malek earlier today



1st definition of T_p

- Based on equiv classes of the $\mathbb{Q}_p^2$-lattice $\mathscr{L}$
- Consider $\mathscr{L} = \left\{ au + bv \in \mathbb{Q}_p^2 \,|\, a, b \in \mathbb{Z}_p \right\}$, where u and v are independent basis vectors in $\mathbb{Q}_p^2$. $\mathscr{L} \sim \mathscr{L}'$ if $\mathscr{L} = c\mathscr{L}'$ for some $c \in \mathbb{Q}_p^\times$
- Assign each **equiv class** to a **vertex** on the tree
- $M \in GL(2,\mathbb{Q}_p)$ acts on $\mathscr{L}$ as matrix multiplication: $M\mathscr{L} = (Mu, Mv)$. Generically $PGL(2,\mathbb{Q}_p)$ takes one equiv class to another.
- But any subgroup conjugate to $PGL(2,\mathbb{Z}_p)$ leaves an equiv class invariant $\Rightarrow T_p$ should really be the homogeneous space $PGL(2,\mathbb{Q}_p)/PGL(2,\mathbb{Z}_p)$, the latter being the max. compact subgroup.
- With vertices, we need edges: $\mathscr{L}$ and $\mathscr{L}'$ are **incident** if $p\mathscr{L} \subset \mathscr{L}' \subset \mathscr{L}$

2nd definition of T_p

[Gubser, Knaute, Parikh, Samberg, Witaszczyk, '16]

Determines norm

- Expand $z = \boxed{p^v} \sum_{m=0} a_m p^m, a_0 \neq 0$

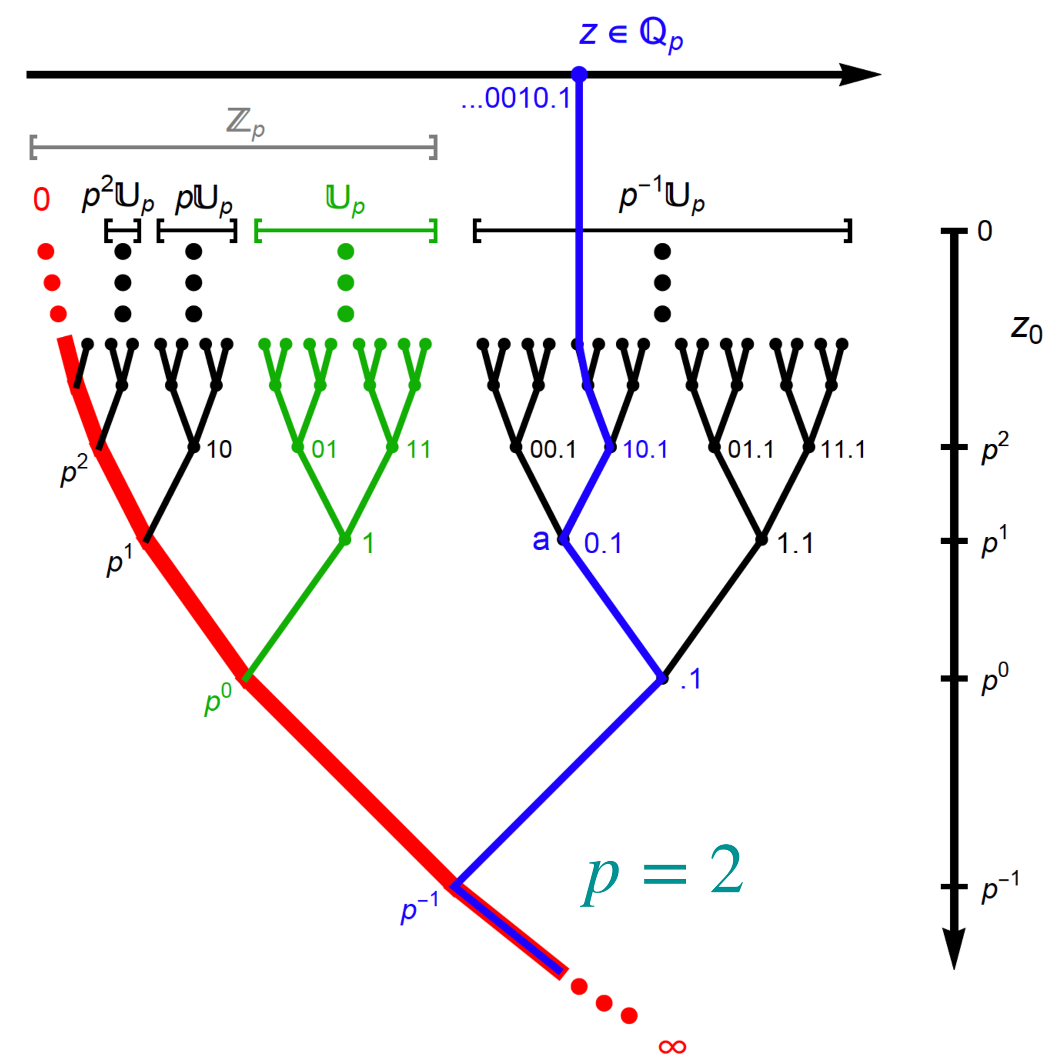
- To move up the tree: choose p-adic digits from right to left, starting from the decimal pt.
  After leaving the red trunk, each node is a rational approximation

- Valence $p + 1$:

  1. Rightmost digit $\neq 0 \rightarrow (p-1)$ choices

  2. Any digit to its left can be zero $\rightarrow p$ choices

- Ring of integers: $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$
  Set of units: $\mathbb{U}_p = \{x \in \mathbb{Q}_p : |x|_p = 1\}$
  Multiplicative subgroup: $\mathbb{Q}_p^\times \equiv \mathbb{Q}_p \backslash \{0\}$



Our results

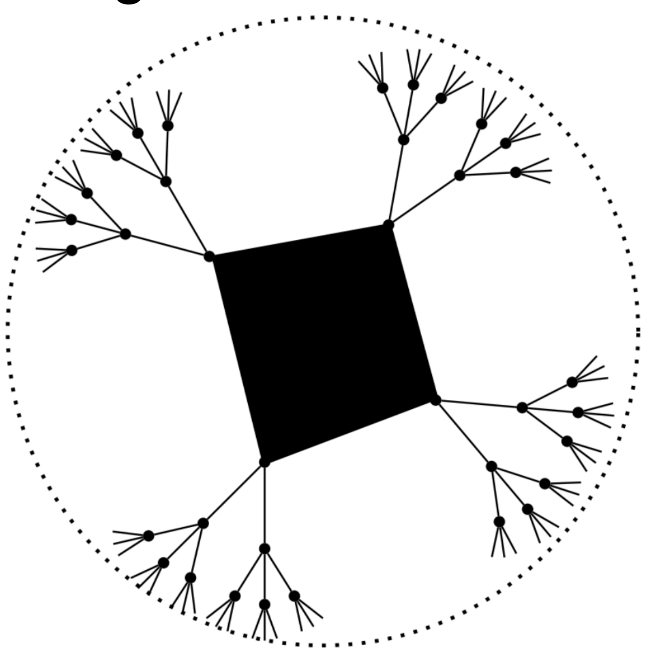
- Computed partition functions for p-adic "thermal AdS" (TAdS) and "BTZ black hole"
- p-adic "genus-1" 1-point function in the semiclassical background of p-adic BTZ
- Set up certain criteria to pinpoint the group representation(s) of $PGL(2,\mathbb{Q}_p)$ for p-adic CFTs

3. p-adic bulk & bdry

Genus > 0 curves

- T^2 identified with the complex lattice $\mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{C}$.
- Usual BTZ is $\mathbb{H}^3/\Gamma$, $\Gamma \subset PSL(2,\mathbb{C})$ a *Schottky group*, generated by $\begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}$, where $q = e^{2\pi i \tau}$
- p-adic BTZ is constructed in the 2nd way: quotient by $\Gamma = \left\langle \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$ where $q \in \mathbb{Q}_p^\times$
- $B \equiv T_p \cup \left(\mathbb{P}^1(\mathbb{Q}_p) \backslash \{0,\infty\} \right)$, where $\{0,\infty\}$ is the limit set $\Rightarrow$ genus-1 bdry
- B/Γ has one regular polygon at the center. The horizon length is $l = \log_p |q|_p > 1$.

$l = 4, p = 3$



- Genus-1: **Tate** (uniformized) elliptic curve
  Genus >1: **Mumford** curve

Axioms of p-adic CFT

[Melzer, '89]

- Has an operator product expansion (OPE) algebra w/ real OPE coeff
- $\mathbb{C}$-valued correlation functions
- Transformation $x \to x' \in P^1(\mathbb{Q}_p)$ is fractional linear:

$$x \to x' = \frac{ax+b}{cx+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL\left(2, \mathbb{Q}_p\right)$$

  Easy to show: $\phi_a'(x')(dx')^{\Delta} = \phi_a(x)(dx)^{\Delta}$, so all fields are primary, dx is Haar measure on $\mathbb{Q}_p$
- Primary operators can have arbitrary dimensions, but the identity operator must have dimension 0.
- 2-pt & 3-pt functions:

$$\Delta_{12} \equiv \Delta_1 + \Delta_2 - \Delta_3$$

$$\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\rangle = \frac{C_{\mathcal{O}_1\mathcal{O}_2}}{|z_{12}|_p^{2\Delta_1}} \qquad \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\rangle = \frac{C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}}{|z_{12}|_p^{\Delta_{12}}|z_{23}|_p^{\Delta_{23}}|z_{31}|_p^{\Delta_{31}}}$$

- Automatically unitary (unlike Archimedean CFT)
- No local conformal algebra or descendants $\Rightarrow$ OPEs are exact. Just the global conformal group $PGL(2,\mathbb{Q}_p)$

p-adic holography

[Gubser, Knaute, Parikh, Samberg, Witaszczyk, '16]

- First established using **propagators & correlators.**
- They look similar!
- Bulk-to-bdry propagators:

ordinary

$$K(z_0, \vec{z}; \vec{x}) = \frac{\zeta_\infty(2\Delta)}{\zeta_\infty(2\Delta - n)} \frac{z_0^\Delta}{(z_0^2 + (\vec{z} - \vec{x})^2)^\Delta}$$

p-adic

$$K(z_0, z; x) = \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta - n)} \frac{|z_0|_p^\Delta}{|(z_0, z - x)|_s^{2\Delta}}$$

- Bulk-to-bulk propagators:

$$G(z_0, \vec{z}; w_0, \vec{w}) = \frac{1}{2\Delta - n} \frac{\zeta_\infty(2\Delta)}{\zeta_\infty(2\Delta - n)} u_\infty^{-\Delta} \times {}_2F_1\left(\Delta, \Delta - n + \frac{1}{2}; 2\Delta - n + 1; -\frac{4}{u_\infty}\right)$$

$$G(z_0, z; w_0, w) = \frac{\zeta_p(2\Delta - n)}{p^\Delta} \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta - n)} u_p^{-\Delta}$$

- 2-pt functions:

$$\langle \mathcal{O}(\vec{x}_1)\mathcal{O}(\vec{x}_2)\rangle_\infty = \eta_\infty L^{n-1}(2\Delta - n) \frac{\zeta_\infty(2\Delta)}{\zeta_\infty(2\Delta - n)} \frac{1}{|\vec{x}_{12}|^{2\Delta}}$$

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_p = \eta_p \frac{p^\Delta}{\zeta_p(2\Delta - n)} \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta - n)} \frac{1}{|x_{12}|_q^{2\Delta}}$$

- 3-pt functions:

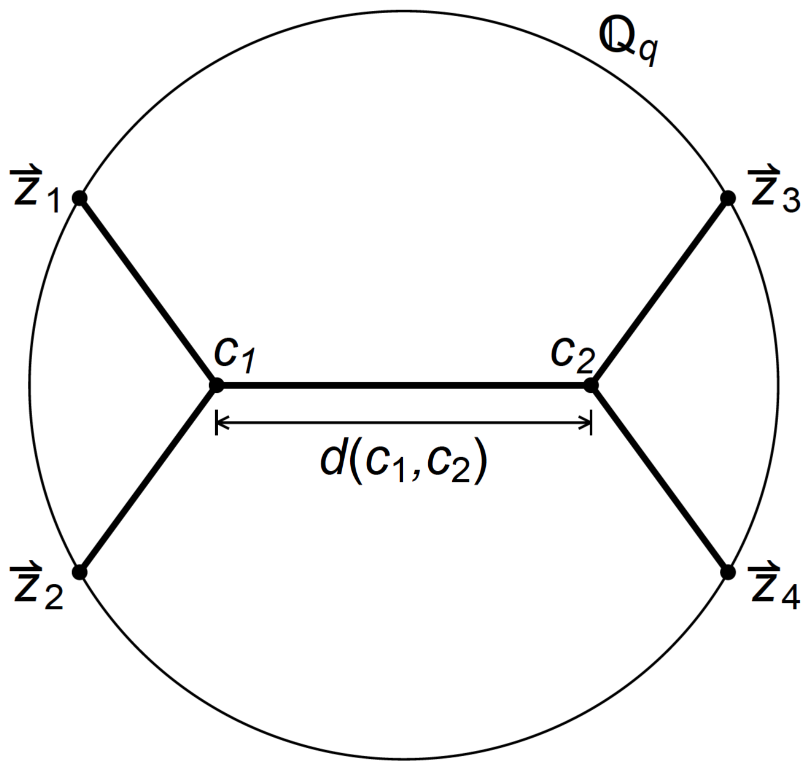
$$\langle \mathcal{O}(\vec{x}_1)\mathcal{O}(\vec{x}_2)\mathcal{O}(\vec{x}_3)\rangle_\infty = -\eta_\infty L^{n-1} g_3 \frac{\zeta_\infty(\Delta)^3 \zeta_\infty(3\Delta - n)}{2\,\zeta_\infty(2\Delta - n)^3} \frac{1}{|\vec{x}_{12}|^\Delta |\vec{x}_{23}|^\Delta |\vec{x}_{13}|^\Delta}$$

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle_p = -\eta_p g_3 \frac{\zeta_p(\Delta)^3 \zeta_p(3\Delta - n)}{\zeta_p(2\Delta - n)^3} \frac{1}{|x_{12}x_{23}x_{13}|_q^\Delta}$$

$$\zeta_\infty(s) \equiv \pi^{-s/2}\Gamma(s/2)$$

p-adic holography

- p-adic 4-pt function much simpler than ordinary one, due to ultrametricity



$$\Longrightarrow |\vec{z}_{13}| = |\vec{z}_{14}| = |\vec{z}_{24}| = |\vec{z}_{23}|$$

- In unramified extension $\mathbb{Q}_{p^n}$, operator T dual to edge length fluctuations on BT tree has 2-pt function $\langle T(z)T(0)\rangle \propto 1/|z|^{2n} \Longrightarrow [T] = n$, as expected for a stress tensor, but still ∄ spin-2 particle…
[Gubser, Heydeman, Jepsen, Marcolli, Parikh, Saberi, Stoica, Trundy, '16]

- Besides these traditional holographic quantities, it has passed checks: Ryu-Takyanagi formula, MERA, etc
[Heydeman, Marcolli, Saberi, Stoica, '16; Hung, Li, Melby-Thompson, '19]

3. Bulk computations

GKP-W dictionary

- In AdS/CFT, For a CFT local operator $\mathcal{O}$, we have

$$Z_{\mathrm{grav}}[\phi_\partial^i(x);\partial M] = \left\langle \exp\left(-\sum_i \int_{\partial M} d^d x \, \boxed{\phi_\partial^i(x)} \mathcal{O}^i(x)\right)\right\rangle_{\mathrm{CFT\ on\ }\partial M}$$

source

with boundary condition $\phi^i(z,x) = z^{d-\Delta}\phi_\partial^i(x) + $ (subleading) as $z \to 0$

[Gubser-Klebanov-Polyakov, '98; Witten, '98]

- By setting $\phi_\partial^i = 0$, the generating functional computes the CFT partition function

- Here $Z_{\mathrm{tree}} = \int \mathcal{D}\phi_a e^{-S_{\mathrm{tree}}[\phi_a]}$

- $S_{\mathrm{tree}}[\phi_a]$ is for massive scalar fields with source. "a" labels vertices

$$S_{\mathrm{tree}}[\phi_a] = \boxed{\sum_{\langle ab\rangle}} \frac{1}{2}(\phi_a - \phi_b)^2 + \sum_a \left(\frac{1}{2}m_p^2\phi_a^2 - \boxed{J_a}\phi_a\right)$$

source

sums over adjacent vertices

GKP-W dictionary

- Linearized EoM: $(\Box + m_p^2)\phi_a = J_a$, where $\Box$ is the lattice/graph Laplacian, a positive definite operator $\Box\phi_a \equiv \sum_{\langle ab \rangle} (\phi_a - \phi_b)$
- Now the partition function is simply $Z_\phi = \frac{1}{\sqrt{\det'(\Box + m_p^2)}}$
- To compute this, we need eigenvalues λ_i's of the Laplacian
- Another way is to use tensor network, making analogy with ordinary diagonal CFTs, to compute it as $\sum_a |q|^{\Delta_a}$

  [Hung, Li, Melby-Thompson, '19]
- Let's first look at "thermal AdS", which is a truncated BT tree.
- All discussions will be on **massless** scalars

p-adic TAdS

- BT tree is homogeneous: can arbitrarily pick the center and assign any vertex with "depth n": # of edges away from the center, whose depth is 0.

- Show $\nexists$ angular modes: take $\phi|_{\partial T_p} \equiv \phi_N = 0$, use the recursion $p(\phi_{N-1} - 0) + (\phi_{N-1} - \phi_{N-2}) = \lambda\phi_{N-1}$ on the fixed ϕ_{N-2} to get $\tilde{\phi}_{N-1} = \phi_{N-1}$. $\blacksquare$

- Consider $J = 0$, from $n = 2$: $p(\phi_{n-1} - \phi_n) + (\phi_{n-1} - \phi_{n-2}) = \lambda\phi_{n-1}$ ★, then

$$\phi_1 = \left(1 - \frac{\lambda}{p+1}\right)\phi_0 \qquad \phi_2 = \left(1 - \frac{2\lambda}{p} + \frac{\lambda^2}{p+p^2}\right)\phi_0$$

Initial condition

From char. eq. → $c_\pm = \left[\frac{1}{2} \pm \frac{p^2 - 1 - \lambda p + \lambda}{2(p+1)\sqrt{(p+1-\lambda)^2 - 4p}}\right]\phi_0$

- Field value $\phi_n = c_+\alpha_+^n + c_-\alpha_-^n$ is a polynomial in eigenvalue λ

Coeff of the highest-degree term: $\dfrac{(-1)^N\phi_0}{p^N + p^{N-1}}$

Coeff of the constant term: ϕ_0

Vieta formula on $\phi_N = 0 \Rightarrow$ **product** of all roots λ_i's of ϕ_N is $p^N + p^{N-1}$

p-adic TAdS

- # of boundary points $\frac{(p+1)p^N - 2}{p-1} \xrightarrow{N\to\infty} \frac{p}{p-1}\left(p^N + p^{N-1}\right)$ diverges
- Recall divergences in ordinary AdS_3/CFT_2:

1. 1-loop determinant of $\Box + m^2$ for a massive scalar on $\mathbb{H}^3$:
   [Giombi-Maloney-Yin, '08]

   UV & IR

   $$\frac{1}{2}\mathrm{Vol}\left(\mathbb{H}^3\right)\int \frac{dt}{t}\frac{e^{-(m^2+1)t}}{(4\pi t)^{3/2}}$$

2. For on-shell Einstein-Hilbert action with constant metric

   $$\frac{1}{16\pi G}\int d^3x\sqrt{g}\,(R-2\Lambda) = \frac{V}{4\pi G l^2}$$

   Introduce a renormalized vol $V_\epsilon(r) = \pi l^3\left(\frac{r^2}{2\epsilon^2} - \frac{1}{2} - \ln\frac{r}{\epsilon}\right)$ [Krasnov, '00]

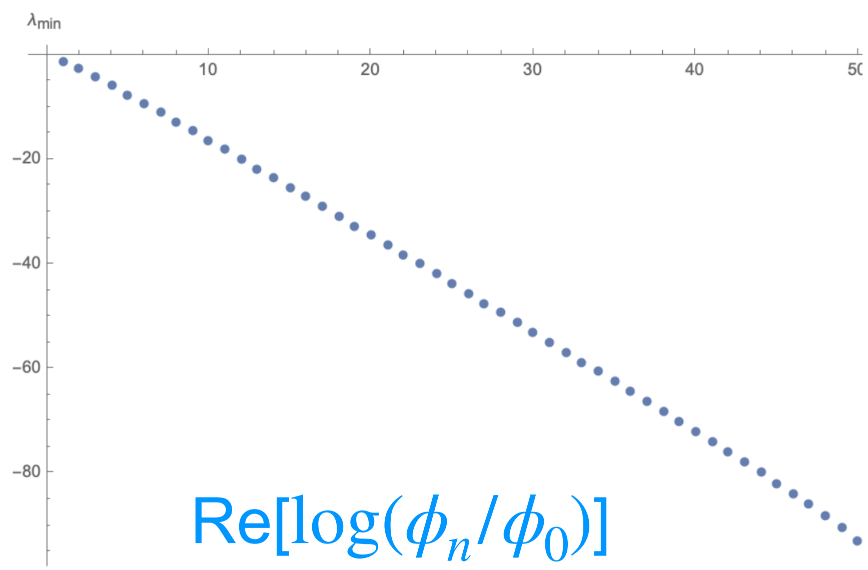
- All can be removed by local counterterms
- In our case, bdry area shows up in e^S instead of in action S, but the volume of BT tree grows exponentially instead of power-law.
- Propose the regularized partition function

$$Z_{\text{tree}} = \left(\frac{p}{p-1}\right)^{1/2}$$

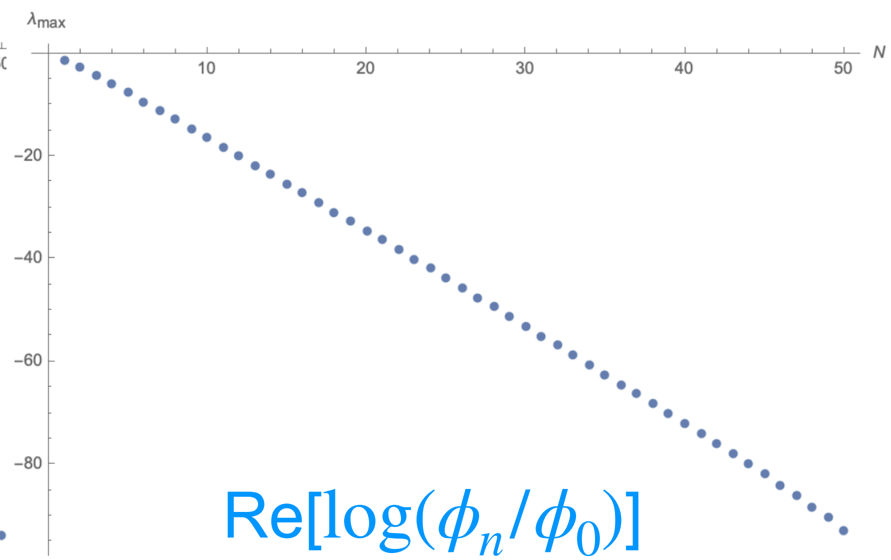
Detailed spectrum

Numerical observations ($N \to \infty$, at fixed p):

1. Upper bound on λ_1 & lower bound on λ_N converge, separately (Newton's method)
2. Decay of field values is almost exponential

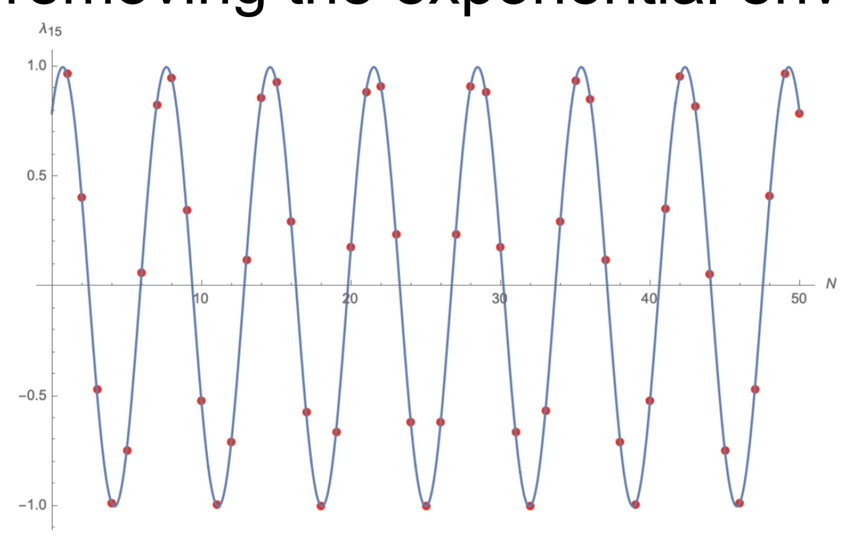


(a) Asymptotics at the smallest eigenvalue.

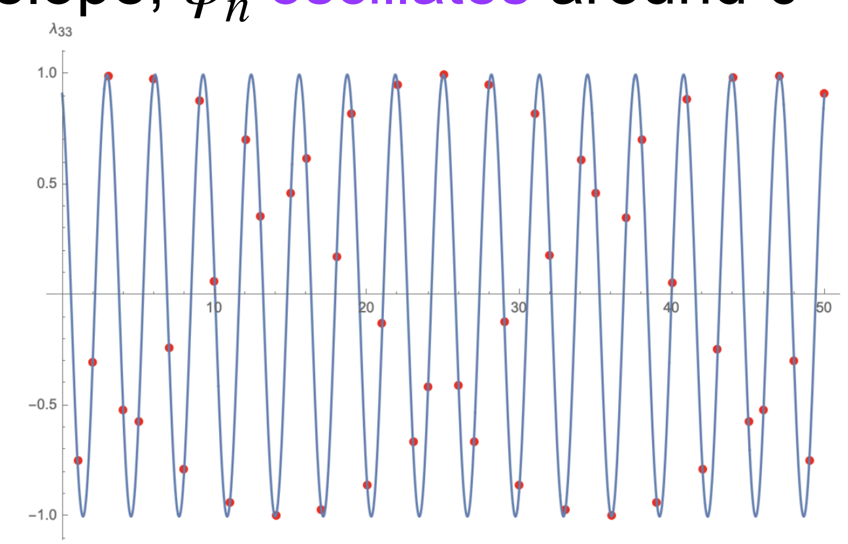


(b) Asymptotics at the largest eigenvalue.

3. After removing the exponential envelope, ϕ_n oscillates around 0



(a) Oscillation of ϕ_n/ϕ_0 at the 15^{th} largest eigenvalue for $p = 239$.



(b) Oscillation of ϕ_n/ϕ_0 at the 33^{th} largest eigenvalue for $p = 239$.

Detailed spectrum

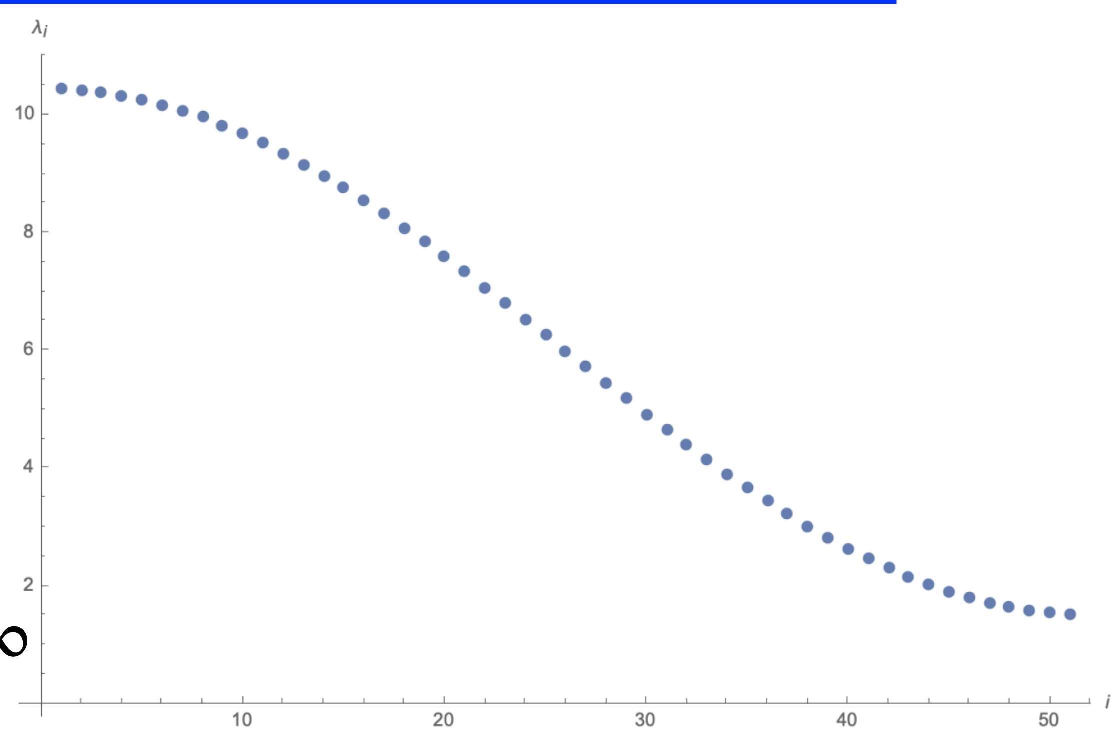
- Ansatz: $$\phi_{n,i} = p^{-n/2}\cos\left(k\,n\,\frac{i-1}{N-1}\pi + \psi\right)\phi_{0,i}$$

  k, ψ: to be determined

- Along with the recursion relation ⇒ $$\lambda_i = p + 1 - 2p^{1/2}\cos\left(k\frac{i-1}{N-1}\pi\right)$$

- How about k?

- Plot all eigenvalues in descending order ($N = 51$)
  ⇒ easy to see that $k = 1$

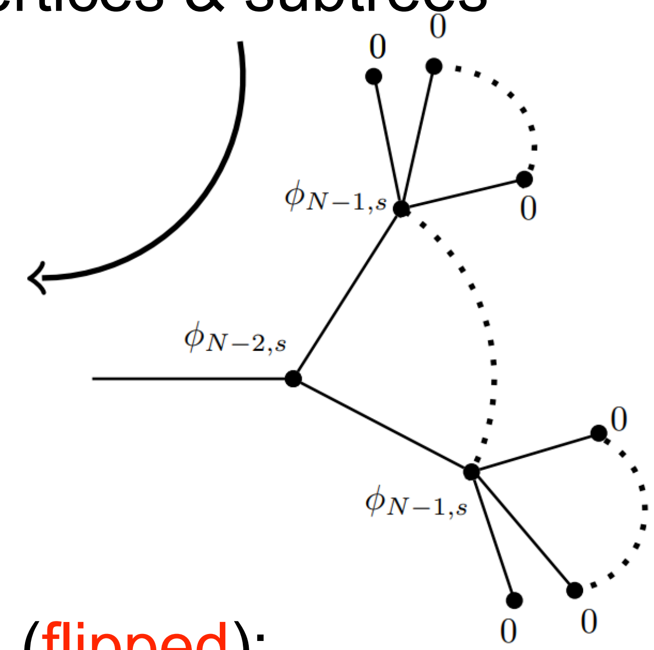


- The blue equation is exact only if the corresponding $\phi_{n,i}$ is for large n, and $N \to \infty$

- Different from the "plane-wave" basis $\epsilon_{\kappa,x}(v) \propto p^{-\kappa d(x,v)}$ in [Heydeman-Marcolli-Saberi-Stoica, '16] $\kappa = 0,1$, $d(x,v)$: distance from bdry pt x
  call it the "**evanescent-wave**" basis

p-adic BTZ

- New feature: field values on horizon (depth 0) can be different — $\phi_{0,0}, \phi_{0,1}, \ldots, \phi_{0,s}, \ldots, \phi_{0,l-1}$, where s labels horizon vertices & subtrees



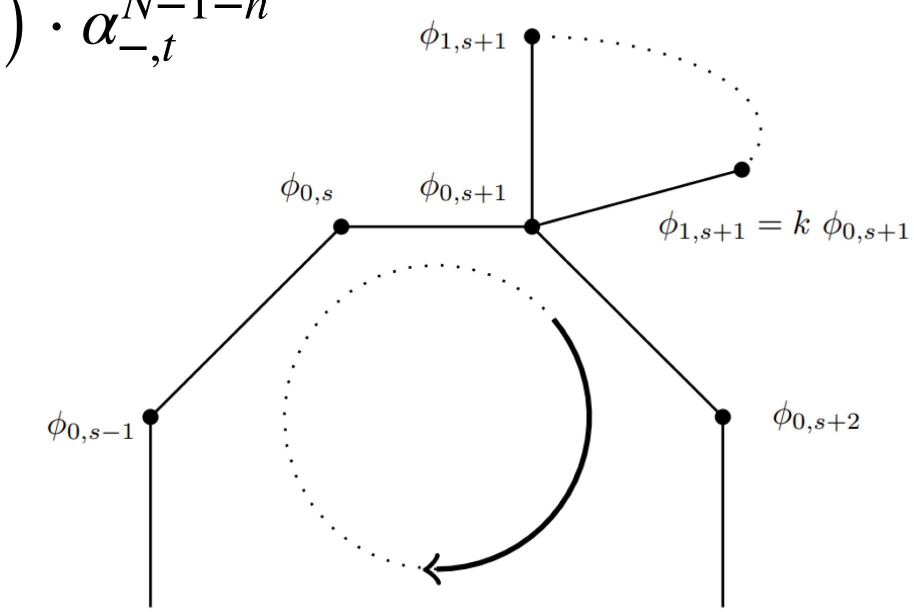
- Boundary (depth N) values vanish, initial condition is $\phi_{N-2,s} = (p+1-\lambda_t)\phi_{N-1,s}$ for $t = 0,\ldots,l-1$
  subscript t in λ_t will be clear soon

- Linear recursion towards the horizon same as before (flipped):
  $\phi_{n-2,s} + (\lambda_t - p - 1)\phi_{n-1,s} + p\phi_{n,s} = 0, \quad 2 \le n \le N-1$ ★ ★

Roots of char. eq.

- Field values: $\phi_{n,s} = c_{+,t}\left(\phi_{N-1,s}\right) \cdot \alpha_{+,t}^{N-1-n} + c_{-,t}\left(\phi_{N-1,s}\right) \cdot \alpha_{-,t}^{N-1-n}$

- Denote ratio b/w field values at depth 1 & on the horizon as $k \equiv \phi_{1,s}/\phi_{0,s}$.
  It is isotropic around the horizon, but still depends on $\alpha_{\pm,t}$ and thus λ_t, so write is as $k_t(\lambda_t)$



p-adic BTZ

- Now the recursion around the horizon is
$$\phi_{0,s+2} - [(p-1)(1-k_t(\lambda_t)) - \lambda_t + 2]\,\phi_{0,s+1} + \phi_{0,s} = 0, \quad s = 0,\ldots,l-1$$ ★ ★ ★

  w/ periodic bdry $\phi_{0,0} = \phi_{0,l}$

- Solution to characteristic eq:
$$r_{\pm,t} = \frac{1}{2}\left\{[(p-1)\,(1-k_t(\lambda_t)) - \lambda_t + 2] \pm \sqrt{[(p-1)(1-k_t(\lambda_t)) - \lambda_t + 2]^2 - 4}\right\}$$

- It must be a root of unity $\Rightarrow$ $k_t(\lambda_t) = 1 - \frac{1}{p-1}\left(2\cos\left(\frac{2\pi t}{l}\right) + \lambda_t - 2\right), \quad t = 0,\ldots,l-1$

  w/ 2-fold degeneracy $k_t(\lambda_t) = k_{l-t}(\lambda_{l-t})$, and t labels **oscillation** modes

- Turns out that the product of all roots for a fixed t is
$$p^N + p^{N-1} + 2\frac{p^{N-1}-1}{p-1} - 2\frac{p^N-1}{p-1}\cos\left(\frac{2\pi t}{l}\right)$$

- Finally, need to multiply contributions from all $t = 1,\ldots,\lceil l/2 \rceil$

- **Key** identity: $\prod_{k=1}^{\beta}\left[2x \pm 2\cos\left(\frac{2\pi k\alpha}{\beta} + \theta\right)\right] = 2\left[T_\beta(x) + (\pm 1)^\beta(-1)^{\alpha\beta+\alpha}\cos(\beta\theta)\right]$

p-adic BTZ

- Expressed in terms of Chebyshev polynomials of the 1st kind:

$$\begin{cases} \sqrt{2}\left(\frac{p^N}{p-1}\right)^{\frac{l}{2}}\left[T_l\left(\frac{p^2+1}{2p}\right)-1\right]^{\frac{1}{2}} & l \text{ even} \\ \sqrt{2}\left(\frac{p^N}{p-1}\right)^{\frac{l}{2}}\left[T_l\left(\frac{p^2+1}{2p}\right)-1\right]^{\frac{1}{2}}\left[\frac{p^{N-1}(p^2+1+2p\cos(\pi/l))}{p-1}\right]^{\frac{1}{2}} & l \text{ odd.} \end{cases}$$

- Diverging p^{lN} as $N \to \infty$ completely differs from: growth of # of bdry pts $l(p-2)(p-1)^{N-1}$, or the growth of total # of pts in the BTZ graph lp^N, so there is no obvious way to regularize, and we keep our results as

$$Z_{\text{BTZ}} = \begin{cases} \left(\frac{p-1}{p^N+1}\right)^{\frac{l}{4}} & l \text{ even} \\ \left(\frac{p-1}{p^N+1}\right)^{\frac{l+1}{4}}\left(\frac{p}{p+1}\right) & l \text{ odd.} \end{cases}$$

- In summary:

1. From bdry to horizon, using recursion ★ ★
2. Go around the horizon, using recursion ★ ★ ★
3. From horizon to bdry, using recursion ★ (just ★ ★ flipped)

Massive scalars

- Can generalize to **massive** scalar (perturbatively in m)

$$m_p^2 = -\frac{1}{\zeta_p(\Delta-1)\zeta_p(-\Delta)} = -(p+1) + 2\sqrt{p}\,\cosh\left[\left(\Delta - \frac{1}{2}\right)\ln\, p\right]$$

  where local zeta function: $\zeta_p(s) \equiv \dfrac{1}{1-p^{-s}}$

  Breitenlohner-Freedman (BF) bound $m_{BF,p}^2 = -1/\zeta_p(-n/2)^2$

- $Z_{\text{tree}}(m \to 0) = \left(p^N + p^{N-1}\right) e^{\frac{Nm^2}{p-1}}$

Unregularized

- $Z_{\text{tree}}(m \to \infty) = \left(p^N + p^{N-1}\right) m^{2N} e^{\frac{N(p+1)}{m^2}}$

- $$Z_{\text{BTZ}}(m \to 0) \approx \begin{cases} \left(1+\frac{lm^2}{2p}\right)^{\frac{1}{2}} \left(\frac{p^{N+1}}{p-1}\right)^{\frac{l}{2}} \left(1+\frac{Nm^2}{(p-1)^2}\right)^{\frac{l}{2}} & l \text{ even} \\ \left(1+\frac{lm^2}{2p}\right)^{\frac{1}{2}} \left(\frac{p^{N+1}}{p-1}\right)^{\frac{l}{2}} \left(1+\frac{Nm^2}{(p-1)^2}\right)^{\frac{l}{2}} \left(A\cos\left(\frac{\pi}{l}\right)+B\right) & l \text{ odd.} \end{cases}$$

- $$Z_{\text{BTZ}}(m \to \infty) \approx \begin{cases} m^{lN-l}\left(1+\frac{N}{m^2}\right)^{\frac{l}{2}} \left(\frac{N(p+1)^2}{2}+\frac{(1-p^2)m^2}{2}\right)^{\frac{1}{2}} & l \text{ even} \\ m^{lN-l}\left(1+\frac{N}{m^2}\right)^{\frac{l}{2}} \left(\frac{N(p+1)^2}{2}+\frac{(1-p^2)m^2}{2}\right)^{\frac{1}{2}} \left(C\cos\left(\frac{\pi}{l}\right)+D\right) & l \text{ odd,} \end{cases}$$

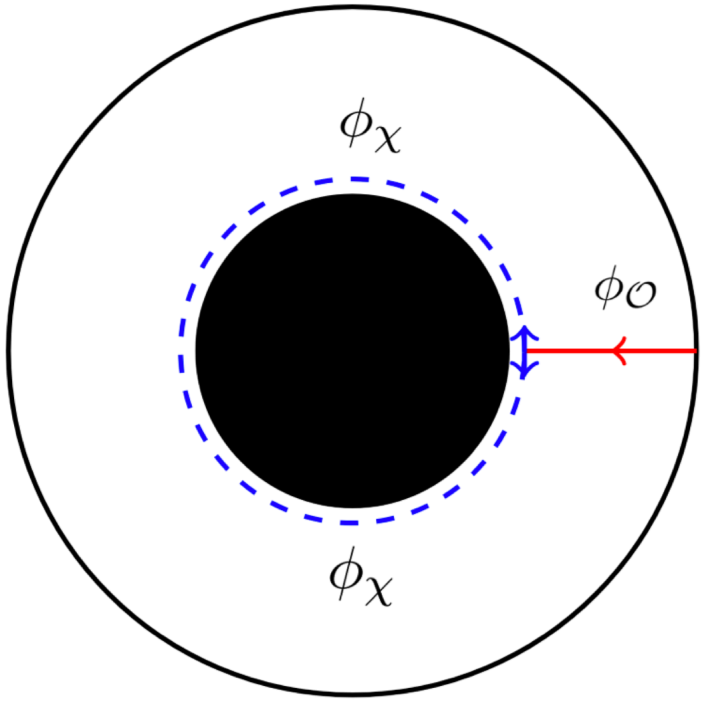
- A, B, C, D are rational functions of m, N, p

4. One-pt function

1-loop Witten diagram

- Modular invariance is crucial in usual 2d CFT: constrains partition functions, spectrum of operator dimensions…
- Torus 1-pt function can be used to estimate high-temperature spectral density weighted by OPE coefficients
- Specifically: $\langle E|\mathcal{O}|E\rangle$
- $|E\rangle$: high-energy state dual to BTZ (semiclassical)
- $\mathcal{O}$, χ: light primary operators dual to light bulk scalars $\phi_{\mathcal{O}}$ and ϕ_χ with energy $E_{\mathcal{O}}, E_\chi \ll c/12$
- $\phi_{\mathcal{O}}$ and ϕ_χ are not conical defects

[Kraus, Maloney, '16]



1-loop Witten diagram

- Averaged 3-pt light-heavy-heavy coefficient $\overline{\langle E|\mathcal{O}|E\rangle} \equiv \frac{\langle E|\mathcal{O}|E\rangle}{\rho(E)}$, taken over all states with energy E

  Denominator: by Cardy formula

  Numerator: $\langle \mathcal{O}\rangle = \mathrm{Tr}_{\mathcal{H}_{S^1}}\, \mathcal{O}\; e^{-\beta H} = \sum_i \langle i|\mathcal{O}|i\rangle\; e^{-\beta E_i}$ S^1: thermal circle

- Asymptotics: exponentially suppressed

$$\overline{\langle E|O|E\rangle} \approx C_{\mathcal{O}\chi\chi} r_+^{\Delta_\mathcal{O}} e^{-2\pi\Delta_\chi r_+}$$

In large r_+ limit

- Can be computed from Witten diagram

$$\overline{\langle E|O|E\rangle} = C_{\mathcal{O}\chi\chi} \int dr dt_E d\phi\; r\; G_{bb}\left(r; \Delta_\chi\right) G_{b\partial}\left(r, t_E, \phi; \Delta_\mathcal{O}\right)$$

- G_{bb} obtained by method of images

$$G_{bb}\left(r, r'\right) = -\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{e^{-\Delta\sigma_n(r,r')}}{1 - e^{-2\sigma_n(r,r')}}$$

$\sigma_n(r, r')$: geodesic distance b/w r & n^{th} image r'

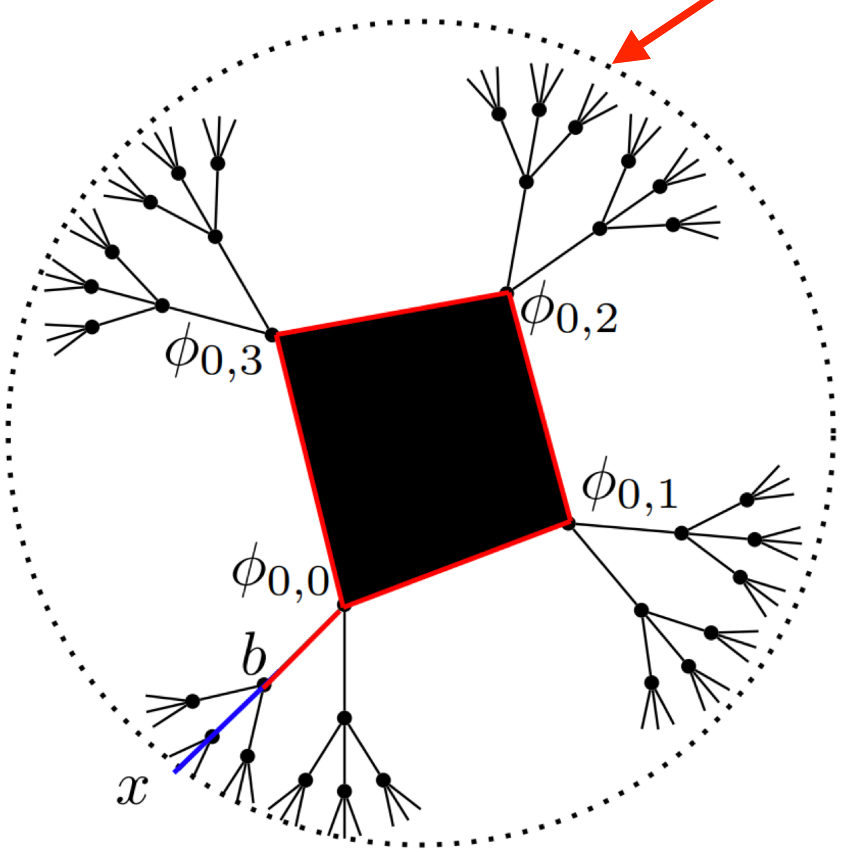
p-adic Witten diagram

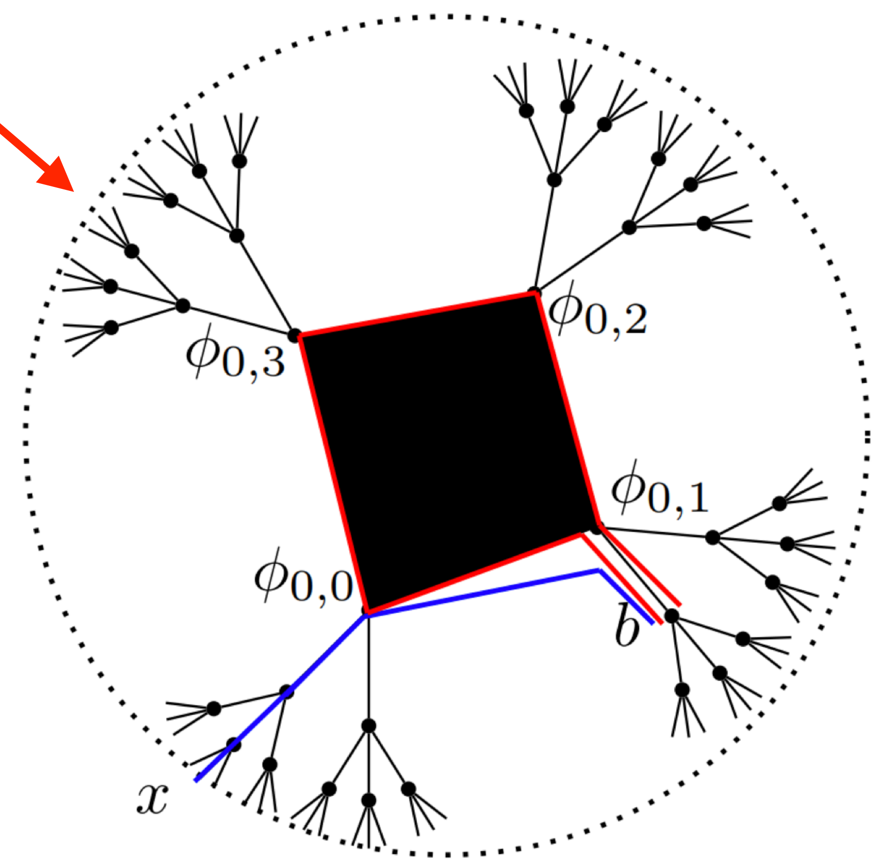
- In p-adic, compute

$$\overline{\langle E|\mathcal{O}|E\rangle} \approx C_{\mathcal{O}\chi\chi} \sum_{(n,h)} d(n,h) G_{bb}(n,h;\Delta_\chi) G_{b\partial}(n,h;x,\Delta_{\mathcal{O}})$$

BT tree reprametrized by (n,h): $n=0$, same subtree

$n \neq 0$, different subtrees





"subways" [Gubser et al., '16]

p-adic Witten diagram

- G_{bb} and $G_{b\partial}$ basically known [Gubser et al., '16; Heydeman et al., '16]

$$G_{bb}(z, z_0; w, w_0) = p^{-\Delta_\chi d(z,z_0;w,w_0)}$$

- In our case,

For χ: $G_{bb}^{\text{renorm}}(n, h) = \frac{2p^{-2\Delta_\chi h}}{p^{\Delta_\chi l} - 1}$ from method of images

For $\mathcal{O}$: $G_{b\partial}(b, x) = p^{-\Delta_{\mathcal{O}} d_{reg}(b,x)} + \frac{2p^{-\Delta_{\mathcal{O}} h}}{p^{\Delta_{\mathcal{O}} l} - 1}$

- Need to regularize the geodesic distance by dictating $d_{reg}(C, x) = 0$ if x is in the subtree rooted at C on horizon [Zabrodin, '89; Heydeman et al., '16]

- $\overline{\langle E|\mathcal{O}|E\rangle} = \overline{\langle E|\mathcal{O}|E\rangle}_{n=0} + \overline{\langle E|\mathcal{O}|E\rangle}_{n\neq 0} = C'_{O\chi\chi}\frac{1}{p^{\Delta_\chi l} - 1} \xrightarrow{l\to\infty} C'_{\mathcal{O}\chi\chi}p^{-\Delta_\chi l}$

- Intuition for no analog of $r_+^{\Delta_{\mathcal{O}}}$: $\mathcal{O}$ unable to "see" the "radius" of p-adic BTZ!

- Expected to be a universal feature for all p-adic CFTs

5. Representations?

Trouble w/ Lie algebras

- It would be great if we can compute 1-pt function using $\langle \mathcal{O} \rangle_\tau = \mathrm{Tr}_{\mathcal{H}}\, \mathcal{O} q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}}$ with $q \equiv e^{2\pi i \tau}$, as in ordinary CFT

- Unfortunately, the exponential map from $PGL(2,\mathbb{Q}_p)$ to "$\mathfrak{pgl}(2,\mathbb{Q}_p)$" doesn't exist: p-adic exponential $\exp(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{n!}$ diverges at identity, since radius of convergence is $|z|_p < p^{-1/(p-1)}$

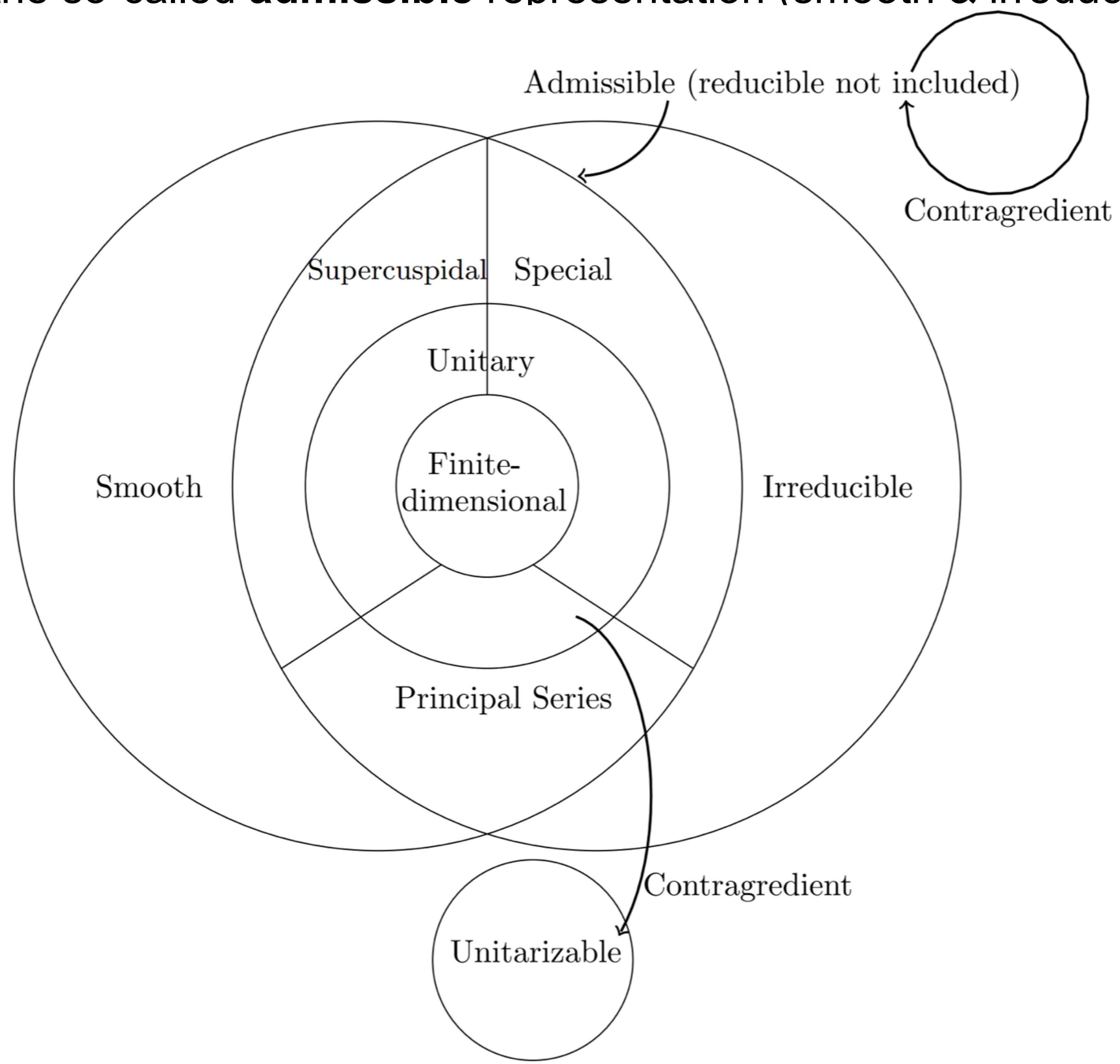
- So Hilbert space $\mathcal{H}$ can't be a rep of algebra, but we still want a group rep. JT or spinors on AdS_2 quantized by group rep of gauge group $SL(2,\mathbb{R}) \times U(1)/\mathbb{Z}$ [Iliesiu, Pufu, Verlinde, Wang, '19] or $\widetilde{SL(2,\mathbb{R})}$ [Kitaev, '17]

- Since all p-adic CFTs are unitary, we want unitary irreps

- All unitary irreps of $PGL(2,\mathbb{Q}_p)$ naturally induces an irrep of $GL(2,\mathbb{Q}_p)$, so we study the latter and then canonically restrict it

Big picture

We want the so-called **admissible** representation (smooth & irreducible)



Narrowing down

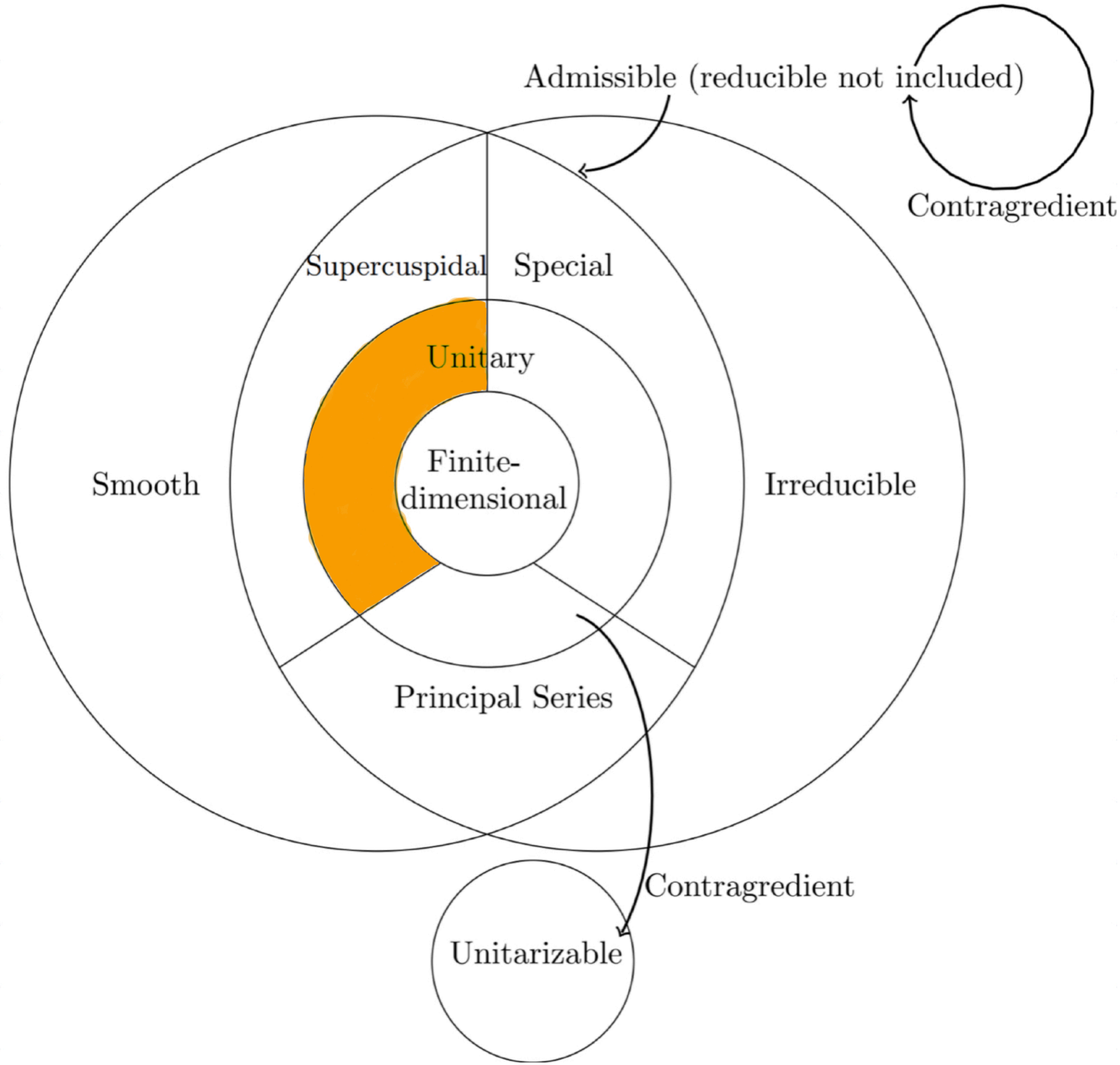
- All finite-dim smooth irreps are trivial: just a 1d $\mathbb{C}$-vector space where $GL(2,\mathbb{Q}_p)$ images act like scalar multiplication

- However, likely that an ensemble of primaries can be viewed as a tensor product of them

- Langlands-like classification of ∞-dim irreps: supercuspidal, principal series, special

- Supercuspidal is desirable, b/c they are the most “native” rep of $GL(2,\mathbb{Q}_p)$: all others can be derived from this, and it has a nicer inner product. Behaves like reps of a compact Lie group.

Big picture, restricted



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- Supercuspidal is desirable, b/c they are the most "native" rep of $GL(2,\mathbb{Q}_p)$: all others can be derived from this, and it has a nicer inner product. Behaves like reps of a compact group.
- Normally, summand in the Virasoro character on torus $\chi(q) = \mathrm{Tr}_{\mathscr{H}} q^{L_0 - \frac{c}{24}}$ can be viewed as a rep of the dilatation transformatic $\begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}$ Schottky
- We want $Z_{p-\mathrm{adic\,CFT}} = \mathrm{Tr}_V\, \pi \left[\begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix} \right]$ from a rep (π, V) of $GL(2,\mathbb{Q}_p)$

Outlook

- Regularization for p-adic BTZ? RG flow? [Hung, Li, Melby-Thompson, ’19; Abdesselam, ’21]
- Detailed spectrum for BTZ?
- Incorporate **true** gravitational fluctuation?
- Other degrees of freedom: gauge fields, susy (fermions), etc
- Connections with spin glass, etc
- Pinpoint the representation(s)

Thank you!